Cutoff Phenomena for Guided Waves in Moving Media

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[redacted] Abstract

This paper treats the propagation of electromagnetic waves in the interior of a waveguide which is filled with a moving medium. The medium is assumed to be homogeneous, isotropic, and lossless, and to move with a constant velocity along the axis of the waveguide. The Maxwell-Minkowski equations for the electromagnetic fields are solved by means of a pair of vector potential functions similar to those frequently used for stationary media. The fields inside the waveguide are derived for both rectangular and cylindrical waveguides.

The well-known cutoff phenomenon for a waveguide is found to be modified in an interesting way when the medium inside the waveguide is moving. The results show that for a slowly moving medium (a medium for which $n\beta < 1$, where n is the index of refraction and β is the velocity of the medium divided by the velocity of light in vacuum) there are two critical frequencies, separating three frequency ranges in each of which there is a different type of propagation. For a high-speed medium $(n\beta > 1)$ it is found that there is no cutoff phenomenon at all, although there is one critical frequency separating two frequency ranges in which the propagation is different.

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Introduction

In this paper we shall consider the propagation of electromagnetic fields in the interior of a rectangular or cylindrical waveguide which is filled with a moving medium. The medium is assumed to be homogeneous, isotropic and lossless, with constitutive parameters μ and ϵ, and to move at uniform velocity along the axis of the waveguide. The waveguide is assumed to have perfectly conducting walls and to be infinitely long. A similar problem has been done by Collier and Tai,[1] under the assumption that the velocity of the medium is much smaller than that of light. In this paper, we shall treat the case where the velocity of the medium can have any value up to the velocity of light. The purpose of the paper is to show how the familiar "cutoff" phenomenon for a waveguide is modified when the medium inside is moving. This effect is not apparent when the velocity of the medium is assumed to be small.

Development of the Theory

The electromagnetic fields inside the waveguide are governed by Maxwell's Equations,

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \qquad (1)$$

$$\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t} + \bar{J} \qquad (2)$$

$$\nabla \cdot \bar{D} = \rho \qquad (3)$$

$$\nabla \cdot \bar{B} = 0 \qquad (4)$$

which, as we know from the special theory of relativity, are valid for any medium, moving or stationary. In Eqs. (1) and (2), $\bar{E}$ and $\bar{H}$ are the electric and magnetic field intensities, $\bar{D}$ and $\bar{B}$ are the electric and magnetic flux densities, and all quantities are measured in a coordinate system which is stationary relative to the walls of the waveguide. [redacted]

The effect of the motion of the medium is to alter the constitutive relations from those that would apply for a stationary medium. The modified constitutive relations for a uniformly moving medium were first derived correctly by Minkowski,[2] and his results have been put into a compact form by Tai.[3] The result is

$$\overline{D} = \epsilon \overline{\overline{\alpha}} \cdot \overline{E} + \overline{\Omega} \times \overline{H} \tag{5}$$

$$\overline{B} = \mu \overline{\overline{\alpha}} \cdot \overline{H} - \overline{\Omega} \times \overline{E} \tag{6}$$

where

$$\overline{\Omega} = \frac{(n^2-1)\beta}{(1-n^2\beta^2)\,c}\,\hat{z} \tag{7}$$

$$n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} \tag{8}$$

$$\beta = \frac{v}{c} \tag{9}$$

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}} = \text{velocity of light in vacuum} \tag{10}$$

$$\overline{\overline{\alpha}} = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

$$a = \frac{1-\beta^2}{1-n^2\beta^2} \tag{12}$$

and μ, ϵ = the permeability and permittivity of the medium (as measured in a frame attached to the medium). Substitution of Eqs.(5) and (6) into (1),(2),(3), and (4) yields the Maxwell-Minkowski equations for the moving istropic medium. They are

$$\nabla \times \overline{E} = -\frac{\partial}{\partial t}\left[\mu \overline{\overline{\alpha}} \cdot \overline{H} - \overline{\Omega} \times \overline{E}\right] \tag{13}$$

$$\nabla \times \overline{H} = \overline{J} + \frac{\partial}{\partial t}\left[\epsilon \overline{\overline{\alpha}} \cdot \overline{E} + \overline{\Omega} \times \overline{H}\right] \tag{14}$$

$$\nabla \cdot \left[\epsilon \overline{\overline{\alpha}} \cdot \overline{E} + \overline{\Omega} \times \overline{H}\right] = \rho \tag{15}$$

$$\nabla \cdot \left[\mu \overline{\overline{\alpha}} \cdot \overline{H} - \overline{\Omega} \times \overline{E}\right] = 0 \tag{16}$$

with $\bar{\bar{J}}$ and ρ considered as sources. For harmonically oscillating fields with a time dependence $e^{-i\omega t}$ these equations may be converted to

$$(\nabla + i\omega\bar{\Omega}) \times \bar{E} = i\omega\mu\,\bar{\bar{\alpha}}.\bar{H} \tag{17}$$

$$(\nabla + i\omega\bar{\Omega}) \times \bar{H} = -i\omega\epsilon\,\bar{\bar{\alpha}}.\bar{E} \tag{18}$$

$$(\nabla + i\omega\bar{\Omega})\cdot(\epsilon\bar{\bar{\alpha}}.\bar{E}) = \rho + \bar{\Omega}\cdot\bar{\bar{J}} \tag{19}$$

$$(\nabla + i\omega\bar{\Omega})\cdot(\mu\bar{\bar{\alpha}}.\bar{H}) = 0. \tag{20}$$

These equations are quite similar to those for a stationary medium except for the substitution of the operator

$$\bar{D}_1 = \nabla + i\omega\bar{\Omega} \tag{21}$$

for the nabla operator ∇ and the appearance of the term $\bar{\Omega}\cdot\bar{\bar{J}}$.

Two types of potential functions will be introduced to describe the electromagnetic fields in the waveguide. Assuming there are no sources, we set $\bar{\bar{J}} = \rho = 0$. Since $\bar{D}_1\cdot\bar{D}_1\times\bar{W} = 0$ for any vector $\bar{W}$,* we may write

$$\mu\,\bar{\bar{\alpha}}\cdot\bar{H}^e = \bar{D}_1 \times \bar{A} \tag{22}$$

where the superscript e denotes field components of the electric type. Substituting Eq. (22) in Eq. (17) gives

$$\bar{D}_1 \times (\bar{E}^e - i\omega\bar{A}) = 0. \tag{23}$$

Also, since $\bar{D}_1 \times \bar{D}_1 U = 0$ for any scalar function U,** we may set

$$\bar{E}^e = i\omega\bar{A} - \bar{D}_1 U \tag{24}$$

If we define another vector function $\bar{A}_1$ such that

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*More precisely, $\bar{D}_1\cdot\bar{D}_1\times\bar{W} = 0$ for any vector $\bar{W}$ whose components have continuous second partial derivatives.

** $\bar{D}_1 \times \bar{D}_1 U = 0$ for any scalar U with continuous second [illegible]

$$\bar{A}_1 = \bar{\bar{\alpha}} \cdot \bar{A} \tag{25}$$

and impose the gauge relation

$$\bar{D}_1 \cdot \bar{A}_1 = i\omega\mu\epsilon a^2 \mathcal{U} \tag{26}$$

between $\bar{A}_1$ and $\mathcal{U}$, it is not difficult to show in terms of cartesian coordinates that $\bar{A}_1$ has to satisfy the equation

$$\left(\bar{D}_a \cdot \bar{D}_1\right) \bar{A}_1 + k^2 a \bar{A}_1 = 0 \tag{27}$$

where

$$\bar{D}_a = \nabla_a + \frac{i\omega}{a} \bar{\Omega} \tag{28}$$

$$\nabla_a = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{1}{a}\frac{\partial}{\partial z} \tag{29}$$

$$k = \omega\sqrt{\mu\epsilon} \tag{30}$$

Equation (27), when written out, reads

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{a}\frac{\partial^2}{\partial z^2} + \frac{2i\omega\Omega}{a}\frac{\partial}{\partial z} - \frac{\omega^2\Omega^2}{a} + k^2 a\right] \bar{A}_1 = 0 \tag{31}$$

The field vectors are then given in terms of $\bar{A}_1$ as

$$\bar{E}^e = i\omega\,\bar{\bar{\alpha}}^{-1} \cdot \bar{A}_1 - \frac{\bar{D}_1\left(\bar{D}_1 \cdot \bar{A}_1\right)}{i\omega\mu\epsilon a^2} \tag{32}$$

$$\bar{H}^e = \frac{1}{\mu}\,\bar{\bar{\alpha}}^{-1} \cdot \left[\bar{D}_1 \times \left(\bar{\bar{\alpha}}^{-1} \cdot \bar{A}_1\right)\right] \tag{33}$$

where $\bar{\bar{\alpha}}^{-1}$ is the inverse of $\bar{\bar{\alpha}}$.

A similar procedure can be followed to find the equations satisfied by the potential functions $\bar{F}$ and V associated with fields $\bar{E}^m$ and $\bar{H}^m$ of magnetic type. The result is given as follows:

$$\bar{E}^m = -\frac{1}{\epsilon}\,\bar{\bar{\alpha}}^{-1} \cdot \left[\bar{D}_1 \times \left(\bar{\bar{\alpha}}^{-1} \cdot \bar{F}_1\right)\right] \tag{34}$$

$$\bar{H}^m = i\omega\,\bar{\bar{\alpha}}^{-1} \cdot \bar{F}_1 - \frac{\bar{D}_1\left(\bar{D}_1 \cdot \bar{F}_1\right)}{i\omega\mu\epsilon a^2} \tag{35}$$

where $\overline{F}$ and $\overline{F}_1$ are related by

$$\overline{F}_1 = \overline{\overline{\alpha}} \cdot \overline{F} \qquad (36)$$

The gauge condition imposed on $\overline{F}_1$ and V is

$$\overline{D}_1 \cdot \overline{F}_1 = i\omega\mu\epsilon a^2 V. \qquad (37)$$

$\overline{F}_1$ has to satisfy the same equation as $\overline{A}_1$.

The field solution in the waveguide can be divided into two basic modes, TE and TM. For TM modes, the field components can be derived from an electric vector potential function $\overline{A} = \hat{z} A$. For TE modes, the field components may be derived from a magnetic vector potential $\overline{F} = \hat{z} F$. For this particular case where $\overline{A}$ and $\overline{F}$ have only one component in the z-direction, we have

$$\overline{A}_1 = \hat{z} A_1 = \overline{\overline{\alpha}} \cdot \overline{A} = \overline{A} = \hat{z} A \qquad (38)$$

$$\overline{F}_1 = \hat{z} F_1 = \overline{\overline{\alpha}} \cdot \overline{F} = \overline{F} = \hat{z} F \qquad (39)$$

The Rectangular Waveguide

The appropriate solutions for A and F which satisfy the boundary conditions for the waveguide configuration shown in Figure 1 are

(PUT FIGURE 1 IN HERE)

to p. 7

$$\bar{A} = \hat{z}A = \hat{z}\, A_0 \sin \frac{m\pi}{X_0} x \sin \frac{\ell\pi}{Y_0} y\, e^{ihz} \tag{40}$$

$$\bar{F} = \hat{z}F = \hat{z}\, F_0 \cos \frac{m\pi}{X_0} x \cos \frac{\ell\pi}{Y_0} y\, e^{ihz} \tag{41}$$

Substituting (40) or (41) into (31) we find

$$h = -\omega\Omega \pm \sqrt{k^2a^2 - k_c^2 a} \tag{42}$$

where

$$k_c^2 = \left(\frac{m\pi}{X_0}\right)^2 + \left(\frac{\ell\pi}{Y_0}\right)^2 \tag{43}$$

Each set of integers m and l corresponds to a given mode which will be designed as the $TM_{m\ell}$ (or $TE_{m\ell}$) modes. The expressions of the electric and magnetic field vectors for the TM modes may be obtained from $\bar{A}$ by means of Eqs. (32) and (33). They are

$$E_x = -\frac{A_0(h+\omega\Omega)}{\omega\mu\epsilon a^2} \frac{m\pi}{X_0} \cos \frac{m\pi}{X_0} x \sin \frac{\ell\pi}{Y_0} y\, e^{ihz} \tag{44}$$

$$E_y = -\frac{A_0(h+\omega\Omega)}{\omega\mu\epsilon a^2} \frac{\ell\pi}{Y_0} \sin \frac{m\pi}{X_0} x \cos \frac{\ell\pi}{Y_0} y\, e^{ihz} \tag{45}$$

$$E_z = A_0\left[i\omega + \frac{(h+\omega\Omega)^2}{i\omega\mu\epsilon a^2}\right] \sin \frac{m\pi}{X_0} x \sin \frac{\ell\pi}{Y_0} y\, e^{ihz} \tag{46}$$

$$H_x = \frac{A_0}{\mu a} \frac{\ell\pi}{Y_0} \sin \frac{m\pi}{X_0} x \cos \frac{\ell\pi}{Y_0} y\, e^{ihz} \tag{47}$$

$$H_y = -\frac{A_0}{\mu a} \frac{m\pi}{X_0} \cos \frac{m\pi}{X_0} x \sin \frac{\ell\pi}{Y_0} y\, e^{ihz} \tag{48}$$

The fields for TE modes may be obtained from $\bar{F}$ by means of Eqs. (34) and (35), with the result

$$E_x = \frac{F_0}{\epsilon a} \frac{\ell\pi}{Y_0} \cos \frac{m\pi}{X_0} x \sin \frac{\ell\pi}{Y_0} y\, e^{ihz} \tag{49}$$

$$E_y = -\frac{F_0}{\epsilon a} \frac{m\pi}{X_0} \sin \frac{m\pi}{X_0} x \cos \frac{\ell\pi}{Y_0} y\, e^{ihz} \tag{50}$$

$$H_x = \frac{F_0(h+\omega\Omega)}{\omega\mu\epsilon a^2} \frac{m\pi}{X_0} \sin \frac{m\pi}{X_0} x \cos \frac{\ell\pi}{Y_0} y\, e^{ihz} \tag{51}$$

$$H_y = \frac{F_0(h+\omega\Omega)}{\omega\mu\epsilon a^2} \frac{\ell\pi}{Y_0} \cos \frac{m\pi}{X_0} x \sin \frac{\ell\pi}{Y_0} y\, e^{ihz} \tag{52}$$

$$H_z \doteq F_0\left[i\omega + \frac{(h+\omega\Omega)^2}{i\omega\mu\epsilon a^2}\right] \cos \frac{m\pi}{X_0} x \cos \frac{\ell\pi}{Y_0} y\, e^{ihz} \tag{53}$$

(PUT FIGURE 2 IN HERE.)

The Cylindrical Waveguide

The proper form of $\overline{A}$ and $\overline{F}$ for the cylindrical waveguide shown in Fig. 2 may be written as

$$\overline{A} = \hat{z} A = \hat{z} A_0 J_m(k_c r) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} m\phi \, e^{ihz} \tag{54}$$

$$\overline{F} = \hat{z} \overline{F} = \hat{z} F_0 J_m(k_c r) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} m\phi \, e^{ihz} \tag{55}$$

where $J_m(k_c r)$ is the Bessel function of integral order m. A and F satisfy (31) which in cylindrical coordinates becomes

$$\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{1}{a}\frac{\partial^2}{\partial z^2} + \frac{2 i \omega \Omega}{a}\frac{\partial}{\partial z} - \frac{\omega^2 \Omega^2}{a} + k^2 a\right]\begin{Bmatrix} A \\ F \end{Bmatrix} = 0 \tag{56}$$

h is given by the same expression shown in (42) while k_c is given as

$$k_c = \frac{\rho_{ml}}{r_0} \quad \text{(TM modes)} \tag{57}$$

$$k_c = \frac{\rho'_{ml}}{r_0} \quad \text{(TE modes)} \tag{58}$$

where ρ_{ml} denotes the roots of the Bessel functions and ρ'_{ml} denotes the roots of the derivatives of the Bessel functions $\frac{\partial J_m(\rho)}{\partial \rho} = 0$. The subscripts m and l denote, respectively, the order of the Bessel function and the index of the root..[4] The complete expressions of the electric and magnetic field vectors for the TM modes are

$$E_r = -\frac{k_c A_0 (h + \omega\Omega)}{\omega \mu \epsilon a^2} \frac{dJ_m(k_c r)}{d(k_c r)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} m\phi \, e^{ihz} \tag{59}$$

$$E_\phi = \pm \frac{A_0 m (h + \omega\Omega)}{\omega \mu \epsilon a^2} \frac{1}{r} J_m(k_c r) \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} m\phi \, e^{ihz} \tag{60}$$

$$E_z = A_0\left[i\omega + \frac{(h+\omega\Omega)^2}{i\omega\mu\epsilon a^2}\right] J_m(k_c r) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} m\phi \, e^{ihz} \tag{61}$$

$$H_r = \mp \frac{m A_0}{\mu a} \frac{1}{r} J_m(k_c r) \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} m\phi \, e^{ihz} \tag{62}$$

$$H_\phi = -\frac{k_c A_0}{\mu a} \frac{dJ_m(k_c r)}{d(k_c r)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} m\phi \, e^{ihz} \tag{63}$$

The field components of the TE modes are:

$$E_r = \pm \frac{F_o m}{\epsilon a} \frac{1}{r} J_m(k_c r) \begin{matrix}\sin\\ \cos\end{matrix} m\phi\, e^{ihz} \tag{64}$$

$$E_\phi = \frac{k_c F_o}{\epsilon a} \frac{d J_m(k_c r)}{d(k_c r)} \begin{matrix}\cos\\ \sin\end{matrix} m\phi\, e^{ihz} \tag{65}$$

$$H_r = -\frac{k_c F_o(h+\omega\Omega)}{\omega\mu\epsilon a^2} \frac{d J_m(k_c r)}{d(k_c r)} \begin{matrix}\cos\\ \sin\end{matrix} m\phi\, e^{ihz} \tag{66}$$

$$H_\phi = \pm \frac{F_o m(h+\omega\Omega)}{\omega\mu\epsilon a^2} \frac{1}{r} J_m(k_c r) \begin{matrix}\sin\\ \cos\end{matrix} m\phi\, e^{ihz} \tag{67}$$

$$H_z = F_o\left[i\omega + \frac{(h+\omega\Omega)^2}{i\omega\mu\epsilon a^2}\right] J_m(k_c r) \begin{matrix}\cos\\ \sin\end{matrix} m\phi\, e^{ihz} \tag{68}$$

The Waveguide Parameters

The formula and the conclusions given in this section apply to both rectangular and cylindrical waveguides (or to a waveguide with any other cross-sectional geometry if k_c is appropriately defined). k_c will assume the value given in (43) for rectangular waveguides and in (57) or (58) for cylindrical waveguides. The propagation constant h is given in (42). When the velocity of the moving medium is small so that $n\beta < 1$, cut off will occur if

$$k^2 a \le k_c^2 \tag{69}$$

and hence the cut off frequency is

$$f_c = \frac{k_c}{2\pi\sqrt{\mu_o\epsilon_o}\sqrt{\frac{n^2(1-\beta^2)}{1-n^2\beta^2}}} \tag{70}$$

when f is less than f_c the fields are attenuated strongly along the guide axis, but unlike an ordinary waveguide below cutoff, there is a phase velocity $v_p = \frac{1}{\Omega}$ in the negative z-direction, for both solutions. When f is slightly greater than f_c there is no attenuation, but the two waves both have phase velocities in the -z-direction (but they have different phase velocities). Finally if f is large enough so that

$$k^2 a^2 - k_c^2 a \ge \omega^2\Omega^2 \tag{71}$$

which can be manipulated to the form

$$f \geq f_+ = \frac{k_c}{2\pi\sqrt{\mu_0\epsilon_0}\sqrt{\frac{n^2-\beta^2}{1-\beta^2}}} \tag{72}$$

then waves can propagate in either direction without attenuation, but again with different phase velocities. If v=0, then β =0, and we have

$$f_+ = f_c = \frac{k_c}{2\pi\sqrt{\mu\epsilon}} \tag{73}$$

which is the usual cutoff frequency in the stationary case.

When $n\beta > 1$, a will be negative while $-\omega\Omega$ is positive. In this case there will be no cutoff phenomenon at all. At low frequencies, the term contributed by the square root in Eq.(42) predominates, so the phase velocities of the two waves are in opposite directions. At higher frequencies $-\omega\Omega$ is always greater than the square root, so both waves have phase velocities in the +z direction. The transition between these two cases occurs at

$$f = f_- = \frac{k_c}{2\pi\sqrt{\mu_0\epsilon_0}\sqrt{\frac{n^2-\beta^2}{1-\beta^2}}} \tag{74}$$

We note that the relation $n\beta>1$ is the condition for Cerenkov radiation in the medium. A summary of these results is presented in Figure 3. (PUT FIGURE 3 IN HERE)

There are an infinite number of modes which can exist in the waveguide but for a given frequency only a finite number of them can propagate freely, assuming the velocity of the medium in the waveguide is small, so that $n\beta<1$. However, if the velocity of the medium is large enough so that $n\beta>1$, then all modes can propagate freely at any frequency. For the case $n\beta<1$, several parameters can be expressed in terms of the cutoff frequency:

$$k_c = 2\pi f_c\sqrt{\mu\epsilon a} \tag{75}$$

$$h = -\omega\Omega \pm a\omega\sqrt{\mu\epsilon}\left[1-\left(\frac{f_c}{f}\right)^2\right]^{1/2}$$ (76

$$= \frac{\omega}{(1-n^2\beta^2)c}\left\{(1-n^2)\beta \pm n(1-\beta^2)\left[1-\left(\frac{f_c}{f}\right)^2\right]^{1/2}\right\} \quad (f \geq f_c)$$ (7

$$h = -\omega\Omega \pm ia\omega_c\sqrt{\mu\epsilon}\left[1-\left(\frac{f}{f_c}\right)^2\right]^{1/2}$$ (7

$$= \frac{1}{(1-n^2\beta^2)c}\left\{\omega(1-n^2)\beta \pm i\omega_c n(1-\beta^2)\left[1-\left(\frac{f}{f_c}\right)^2\right]^{1/2}\right\} \quad (f \leq f_c)$$ (

where $\omega_c = 2\pi f_c$. For $f > f_c$, the guide phase velocity and guide wavelength are respectively

$$v_g = \frac{\omega}{h} = (1-n^2\beta^2)c \Big/ \left\{(1-n^2)\beta \pm n(1-\beta^2)\left[1-\left(\frac{f_c}{f}\right)^2\right]^{1/2}\right\}$$ (

$$\lambda_g = \frac{2\pi}{h} = (1-n^2\beta^2)\lambda_0 \Big/ \left\{(1-n^2)\beta \pm n(1-\beta^2)\left[1-\left(\frac{f_c}{f}\right)^2\right]^{1/2}\right\}$$ (8

where $c = 1/\sqrt{\mu_0\epsilon_0}$ and λ_0 is the free space wavelength. The $TM_{m\ell}$ characteristic wave impedance for the wave travelling in the +z direction is

$$Z^{TM}_{m\ell} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} \quad \text{(rectangular waveguides)}$$ (82

$$= \frac{E_r}{H_\phi} = -\frac{E_\phi}{H_r} \quad \text{(cylindrical waveguides)}$$ (83

$$= \frac{h+\omega\Omega}{\omega\epsilon a} = \frac{\sqrt{k^2a^2 - k_c^2a}}{\omega\epsilon a} \quad (n\beta<1 \text{ or } n\beta>1)$$ (84

$$= \eta\left[1-\left(\frac{f_c}{f}\right)^2\right]^{1/2} \quad (n\beta<1 \text{ and } f>f_c)$$ (85

$$= i\frac{\omega_c}{\omega}\eta\left[1-\left(\frac{f}{f_c}\right)^2\right]^{1/2} \quad (n\beta<1 \text{ and } f<f_c)$$ (86

where $\eta = \sqrt{\frac{\mu}{\epsilon}}$ is the intrinsic impedance of the medium. Similarly, the TE characteristic wave impedance (for the +z wave) is also given by equations (82) and (83), which for this case yield

$$Z_{m\ell}^{TE} = \frac{\omega\mu a}{h+\omega\Omega} = \frac{\omega\mu a}{\sqrt{k^2a^2 - k_c^2 a}} \quad (n\beta<1 \text{ or } n\beta>1) \tag{87}$$

$$= \eta\left[1-\left(\frac{f_c}{f}\right)^2\right]^{-1/2} \quad (n\beta<1 \text{ and } f>f_c) \tag{88}$$

$$= -i\frac{\omega}{\omega_c}\eta\left[1-\left(\frac{f}{f_c}\right)^2\right]^{-1/2} \quad (n\beta<1 \text{ and } f<f_c) \tag{89}$$

It is interesting to note that the product $Z_{m\ell}^{TM} Z_{m\ell}^{TE} = \eta^2 = \frac{\mu}{\epsilon}$ at all frequencies and for all velocities of the medium. $Z_{m\ell}^{TM}$ and $Z_{m\ell}^{TE}$ as given in (85) and (86) or (88) and (89) are of the same form as when the medium in the waveguide is stationary. The power flow in the rectangular waveguide for TM modes is

$$P = Re\,\frac{1}{2}\int_0^{x_0}\int_0^{y_0} \overline{E}\times\overline{H}^*\cdot\overline{dS} = \frac{x_0 y_0}{2\epsilon_{om}\epsilon_{o\ell}}|A_0|^2 k_c^2 \frac{h+\omega\Omega}{\omega\mu^2\epsilon a^3} \quad (n\beta<1 \text{ or } n\beta>1) \tag{90}$$

$$= \pm\frac{x_0 y_0 k_c^2 |A_0|^2}{2\epsilon_{om}\epsilon_{o\ell}} \frac{\left[1-(f_c/f)^2\right]^{1/2}}{a^2\mu\sqrt{\mu\epsilon}} \quad (n\beta<1) \tag{91}$$

where $\epsilon_{o\ell}$ is defined as equal to 1 when $\ell = 0$ and equal to 2 when $\ell > 0$. For TE modes it is

$$P = \frac{x_0 y_0 k_c^2 |F_0|^2}{2\epsilon_{om}\epsilon_{o\ell}} \cdot \frac{h+\omega\Omega}{\omega\mu\epsilon^2 a^3} \quad (n\beta<1 \text{ or } n\beta>1) \tag{92}$$

$$= \pm\frac{x_0 y_0 k_c^2 |F_0|^2}{2\epsilon_{om}\epsilon_{o\ell}} \frac{\left[1-(f_c/f)^2\right]^{1/2}}{a^2\epsilon\sqrt{\mu\epsilon}} \quad (n\beta<1) \tag{93}$$

In cylindrical waveguides the corresponding expressions are

$$P = \frac{\pi r_o^2 |A_o|^2}{2\epsilon_{om}} \left[\frac{d}{dr} J_m(k_c r) \Big|_{r=r_o} \right]^2 \frac{h+\omega\Omega}{\omega\mu^2\epsilon a^3} \quad (n\beta<1 \text{ or } n\beta>1) \tag{94}$$

$$= \pm \frac{\pi r_o^2 |A_o|^2}{2\epsilon_{om}} \left[\frac{d}{dr} J_m(k_c r) \Big|_{r=r_o} \right]^2 \frac{\left[1-(f_c/f)^2\right]^{1/2}}{a^2\mu\sqrt{\mu\epsilon}} \quad (n\beta<1) \tag{95}$$

for TM modes and

$$P = \frac{\pi k_c^2 r_o^2 |F_o|^2}{2\epsilon_{om}} \left[1 - \frac{m^2}{k_c^2 r_o^2}\right] J_m^2(k_c r_o) \frac{h+\omega\Omega}{\omega\mu\epsilon^2 a^3} \quad (n\beta<1 \text{ or } n\beta>1) \tag{96}$$

$$= \pm \frac{\pi k_c^2 r_o^2 |F_o|^2}{2\epsilon_{om}} \left[1 - \frac{m^2}{k_c^2 r_o^2}\right] J_m^2(k_c r_o) \frac{\left[1-(f_c/f)^2\right]^{1/2}}{a^2\epsilon\sqrt{\mu\epsilon}} \quad (n\beta<1) \quad (9$$

for TE modes. Although the phase velocities are as shown in Fig 3, the power flow for the two waves in the guide are in each case of the same magnitude and in opposite directions.

When the velocity $\bar{v}$ approaches zero or when the constitutive parameters of the medium equal those of free space Ω will approach zero and a will approach one; all the results obtained then reduce to the familiar ones for the medium at rest.

Conclusions

From the Maxwell-Minkowski equations for the electromagnetic field in a moving medium, it has been shown how the electromagnetic fields may be constructed from a pair of vector potential functions $\overline{A}$ and $\overline{F}$, which are derived using a technique similar to that commonly used for stationary media. The solutions for $\overline{A}$ and $\overline{F}$ appropriate to a rectangular and a cylindrical waveguide have been given, as well as the formulas for the fields. These results show the dependence of the fields on the velocity of the medium inside the waveguide.

The propagation constant for the fields in the waveguide has been examined to determine how the motion of the medium affects the cutoff behavior. It was found that the well-known cutoff frequency for a waveguide is modified when the medium inside is moving. For $n\beta < 1$, corresponding to a slowly moving medium, there are two critical frequencies f_c and f_+. For $f < f_c$, the fields are attenuated, as in a conventional waveguide, but also have a phase velocity, unlike a conventional waveguide. For $f_c < f < f_+$, the fields are unattenuated but all fields have a phase velocity in the -z direction. For $f > f_+$, waves may travel unattenuated in either direction in the waveguide, but with a different phase velocity in each direction. For $n\beta > 1$, corresponding to the case of Cerenkov radiation, there is one critical frequency f_-. For $f < f_-$, waves may travel with a phase velocity in either direction. For $f > f_-$, all solutions have a phase velocity in the +z direction. Also for $n\beta > 1$, there is no cutoff phenomenon in the usual sense. Waves may propagate unattenuated at any frequency.

Finally, some formulas for the waveguide characteristic impedance and for the power flow in a waveguide filled with a moving medium have been given.

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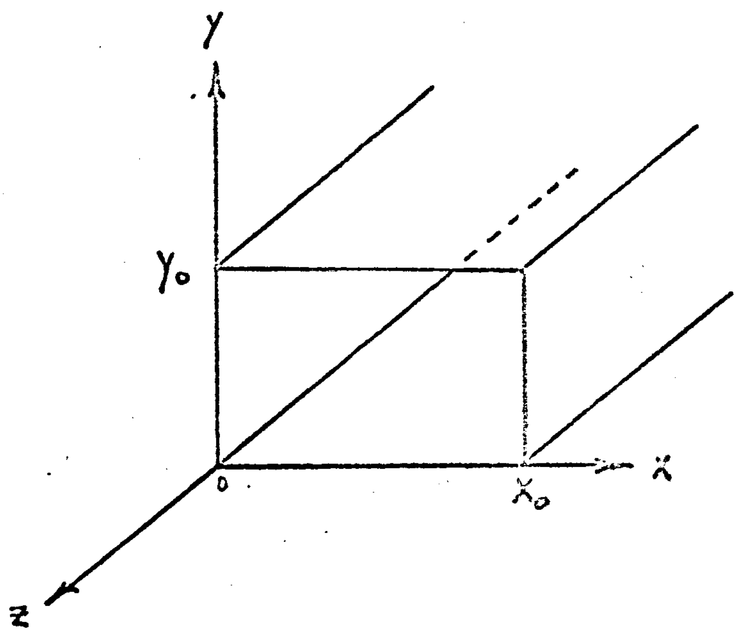
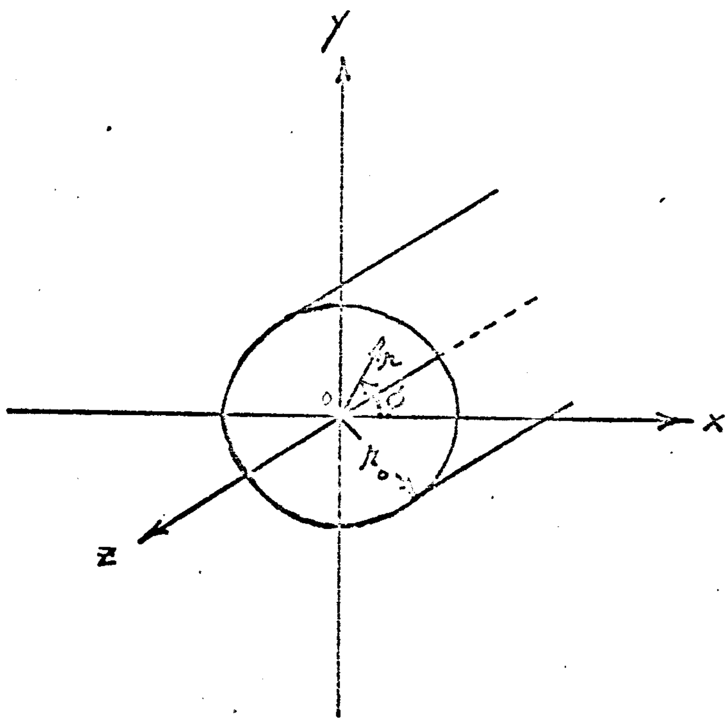


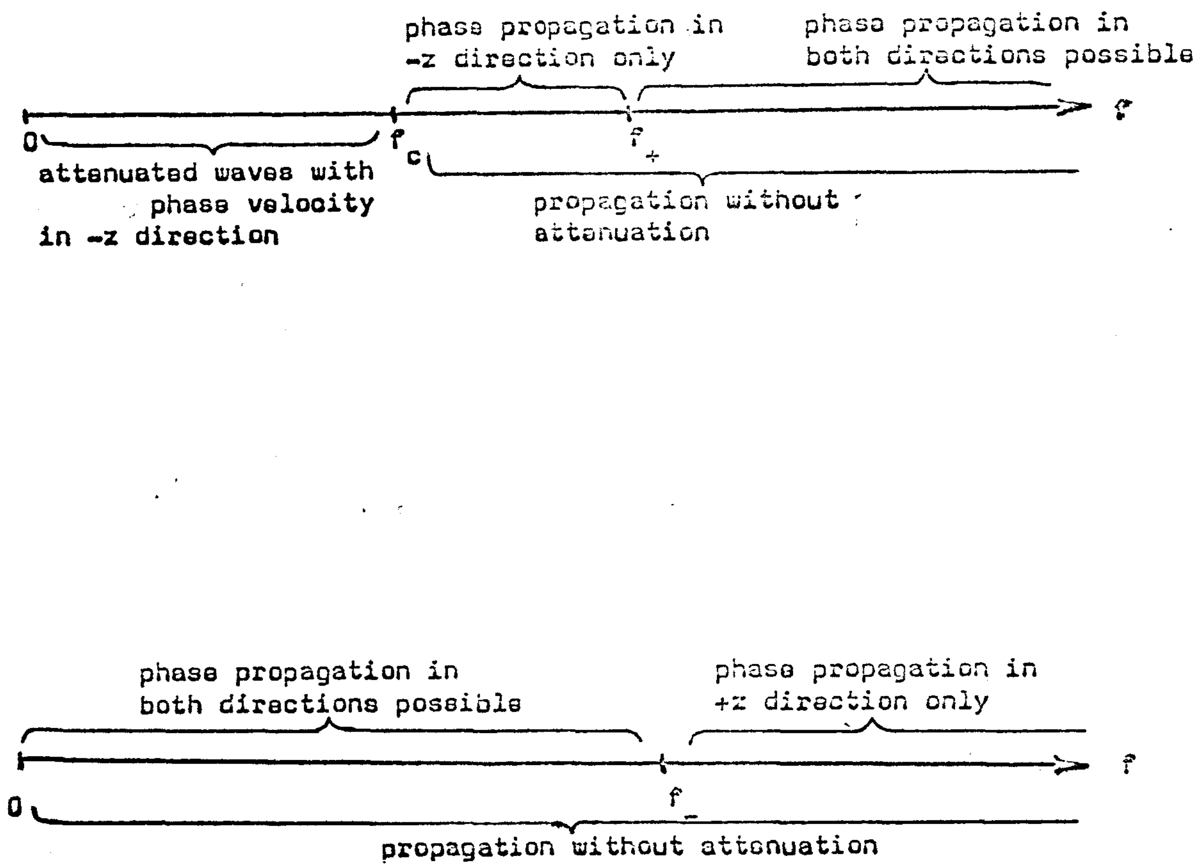
The Rectangular Waveguide

Figure 1



The Cylindrical Waveguide

Figure 2



Frequency Ranges for Wave Propagation in the Waveguide,
with the medium moving in the **+z** direction

Figure 3